Saginaw Valley State University
2006 Math Olympics – Level II Solutions

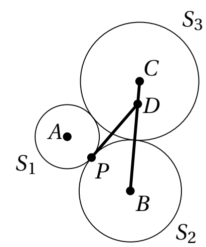
1. How many zeros are at the end of 79!

   (Recall, $n!$ is the product $1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.)

   (a) none **(b)** 18 **(c)** 15 **(d)** 7 **(e)** None of the above

   SOLUTION **(b)**: A zero comes from a factor of 10. A factor of 10 comes from a factor of 2 and a factor of 5. There are plenty of factors of 2 (every other number has at least one), so the only question is how many factors of 5 are there. Counting by 5's there are 15 multiples of 5, but 3 of these, 25, 50, 75, have 2 factors of 5. This makes 18 total.

2. Three circles, S_1, S_2 and S_3, of different radii with centers A, B, C, respectively, are drawn so that any two touch each other. The point P is the intersection of the smaller circles S_1 and S_2. The common tangent to S_1 and S_2 is extended from P and meets BC at D. If the radii of S_1, S_2 and S_3 are $2, 3$, and 10, respectively, find the length of CD.



   (a) 6 **(b)** $\frac{26}{5}$ **(c)** $\frac{13}{3}$ **(d)** $\frac{13}{5}$ **(e)** None of the above

   SOLUTION **(b)**: The sides of the triangle ABC are $(13, 12, 5)$, hence by the converse of the Pythagorean theorem, the angle BAC is $90°$. Since PD is a tangent to the circles S_1 and S_2, the angle DPB is also $90°$. Now, by the similarity of the triangles ABC and PBD, we have $BD = (BC)(PB)/(AB) = (13)(3)/(5) = \dfrac{39}{5}$. Thus $CD = BC - BD = 13 - \dfrac{39}{5} = \dfrac{26}{5}$.

3. Let $n = \sqrt{6+\sqrt{11}} + \sqrt{6-\sqrt{11}} - \sqrt{22}$. Then

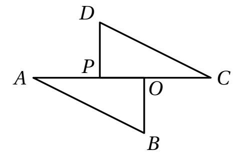
   (a) $n \geq 1$ **(b)** $0 < n < 1$ **(c)** $n = 0$ **(d)** $-1 < n < 0$ **(e)** $n \leq -1$

   SOLUTION **(c)**: We need to compare the numbers $a = \sqrt{6+\sqrt{11}} + \sqrt{6-\sqrt{11}}$ and $b = \sqrt{22}$. Since both numbers are obviously positive, we can compare their squares instead.

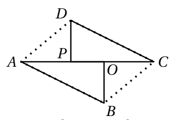
$$
\begin{aligned}
a^2 &= \left(\sqrt{6+\sqrt{11}} + \sqrt{6-\sqrt{11}}\right)^2 \\
&= 6 + \sqrt{11} + 2\sqrt{6+\sqrt{11}}\sqrt{6-\sqrt{11}} + 6 - \sqrt{11} \\
&= 12 + 2\sqrt{\left(6+\sqrt{11}\right)\left(6-\sqrt{11}\right)} \\
&= 12 + 2\sqrt{36-11} \\
&= 22
\end{aligned}
$$

   Therefore $a^2 = 22 = b^2$, therefore $a = b$ and $n = 0$.

4. In the figure two congruent 30°-60° right triangles, AOB and CPD are drawn so that AO is opposite to the 60° angle in AOB and so that the points O and P trisect the segment $\overline{AC}$. If the length of $\overline{DP}$ is one unit, find the perimeter of the quadrilateral $ABCD$ (not drawn).



(a) $2(2+\sqrt{3})$ **(b)** $2(3+\sqrt{2})$ **(c)** $4+2\sqrt{2}$ **(d)** $4+\sqrt{7}$ **(e)** None of the above



SOLUTION **(d)**: Since the triangles AOB and CPD are congruent and the points O and P trisect the segment $\overline{AC}$, the side CP is opposite to the 60° angle in the triangle CPD. Recall that the ratios of the sides of a 30°-60°-right triangle are $(1 : \sqrt{3} : 2)$. So, the side AB is 2 units and CP is $\sqrt{3}$ units. Hnce $AP = \sqrt{3}/2$. Therefore, the hypotenuse AD in the right triangle APD is $\sqrt{7}/2$ units. Thus the perimeter of $ABCD$ is $2(2+\sqrt{7}/2)$.

5. Find the number of 6-digit numbers that use only the digits 1 and 9, but do not have two 1's next to each other.

(a) 19 **(b)** 21 **(c)** 49 **(d)** 14 **(e)** None of the above

SOLUTION **(b)**: You can reason as follows:

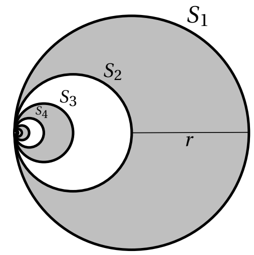
- The number of such 6-digit numbers with no 1's is 1.
- The number of such 6-digit numbers with exactly one 1 is 6.
- The number of such 6-digit numbers with exactly two 1's can be found as follows: If the first 1 is the first digit, the second 1 can be the third, fourth, fifth, or sixth digit. If the first 1 is the second digit, the second 1 can be the fourth, fifth, or sixth digit. If the first 1 is the third digit, the second 1 can be the fifth or sixth digit; and if the first 1 is the fourth digit, the second 1 must be the sixth digit. The first 1 can't be the fifth digit. So there are $1+2+3+4=10$ such 6-digit numbers.
- The number of such 6-digit numbers with exactly three 1's can be found in the same way. If the first 1 is the first digit, and the second 1 is the third digit, the third 1 can be the fifth or sixth digit. If the first 1 is the first digit, and the second 1 is the fourth digit, the third 1 must be the sixth digit It is impossible for the second 1 to be the fifth digit. If the first 1 is the second digit, the second must be the fourth digit and the third must be the sixth digit. It is impossible for the first 1 to be later than the second digit, so there are only these 4 choices.
- It is impossible for there to be more than three 1's without having two of them next to each other so there are $1+6+10+4=21$.

6. The sum of the solutions of the equation $\log_2\left(x^2-8x-8\right)=0$ is

(a) 9 **(b)** 10 **(c)** -8 **(d)** 8 **(e)** None of the above

SOLUTION **(d)**: Taking the exponential function of base 2 of both sides, we get $x^2-8x-8=1$. That's a quadratic equation with leading coefficient 1, so the sum of the two solutions is the opposite of the coefficient for x, which is 8.

7. Let $S_1, S_2, S_3, \ldots$ be a sequence of nested circles such that S_{i+1} is inside S_i and the radius of S_{i+1} is half of that of S_i. Let r denote the radius of S_1. For $i = 1, 3, 5, \ldots$ (i.e. for odd i's), the region inside S_i and outside S_{i+1} is shaded. Find the sum of the shaded areas.



(a) $\dfrac{3\pi r^2}{4}$

(b) $\dfrac{\pi r^2}{4}$ **(c)** $\dfrac{4\pi r^2}{5}$ **(d)** $3\pi r^2$ **(e)** None of the above

SOLUTION **(c)**: The circle S_k has radius $\dfrac{r}{2^{k-1}}$ and its area is $\dfrac{\pi r^2}{2^{2(k-1)}}$. Thus the shaded region has area equal to the sum of the infinite series

$$\pi r^2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2^{2(k-1)}} = \pi r^2 \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l}} = \pi r^2 \sum_{l=0}^{\infty} \left(\frac{-1}{4}\right)^l.$$

Clearly, the last infinite sum is a geometric series with first term 1 and ratio $-1/4$; hence the shaded area converges to $\pi r^2 \dfrac{1}{1+\frac{1}{4}} = \dfrac{4\pi r^2}{5}$.

8. For which values of a is $(\log_a 7)(\log_7 6) = \ln 36$?

(a) $\frac{1}{e}$ **(b)** 6 **(c)** $\sqrt{e}$ **(d)** 2 **(e)** None of the above

SOLUTION **(c)**: According to the base change formula,

$$(\log_a 7)(\log_7 6) = \frac{\ln 7}{\ln a} \cdot \frac{\ln 6}{\ln 7} = \frac{\ln 6}{\ln a}$$

which means that

$$\ln a = \frac{\ln 6}{\ln 36} = \frac{\ln 6}{2\ln 6} = \frac{1}{2}$$

which means $a = e^{1/2}$.

9. For which x is $2\sin^{-1}\left(\sin\dfrac{x}{2}\right) = x$?

(a) For all real x **(b)** For $-1 \le x \le 1$ **(c)** For $-\pi \le x \le \pi$

(d) For $-\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}$ **(e)** None of the above

SOLUTION **(c)**: The range of $\sin^{-1}$ is the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore

$$\sin^{-1}(\sin\frac{x}{2}) = \frac{x}{2} \text{ if } -\frac{\pi}{2} \le \frac{x}{2} \le \frac{\pi}{2}.$$

So $2\sin^{-1}(\sin\frac{x}{2}) = x$ if $-\pi \le x \le \pi$.

10. At exactly 3:22, what is the angle between the hour hand and the minute hand of a clock? (Assume that both hands rotate in smooth continuous motion without jumps.)

    (a) $35°$ **(b)** $7°$ **(c)** $31°$ **(d)** $42°$ **(e)** None of the above

    SOLUTION **(c)**: Each minute, the hour hand rotates by $1/2°$, while minute hand rotates by $6°$. Hence at 3:22, the angle between the hour hand and 3 o'clock is $11°$, while the angle between the minute hand and 3 o'clock is $6(22-15) = 42°$. Therefore the angle between hour hand and minute hand at 3:22 is $42° - 11° = 31°$.

11. For which real values of p does the graph of $y = x^2 - 2px + p + 1$ have exactly one x-intercept?

    (a) No such p exists **(b)** $p = 0$ **(c)** $p = \dfrac{1+\sqrt{5}}{2}$ only

    (d) $p = \frac{1}{2}$ **(e)** None of the above

    SOLUTION **(e)**: In order for the graph to have exactly one intercept, the polynomial $x^2 - 2px + p + 1$ would have to be a perfect square. Therefore $p^2 = p + 1$, or $p^2 - p - 1 = 0$, which means that

$$p = \frac{1 \pm \sqrt{5}}{2}.$$

12. A committee of 4 people is to be randomly chosen from a group of 7 people, including Mr. and Mrs. Smith. What is the probability that a committee will not contain both Mr. Smith and Mrs. Smith?

    (a) $\frac{5}{7}$ **(b)** $\frac{1}{3}$ **(c)** $\frac{4}{7}$ **(d)** $\frac{2}{3}$ **(e)** None of the above

    SOLUTION **(a)**: The number of possible committees is $\frac{7\cdot6\cdot5\cdot4}{4\cdot3\cdot2} = 35$. The number of committees that contain both Mr. and Mrs. Smith is given by the number of ways to choose the other two people on the committee, which is $\frac{5\cdot4}{2} = 10$. So the number of committees that do not contain both Mr. and Mrs. Smith is $35 - 10 = 25$. So the probability that a committee will not contain both Mr. and Mrs Smith is $\frac{25}{35} = \frac{5}{7}$.

13. $\cos\left(2\cos^{-1}\left(-\frac{12}{13}\right)\right) =$

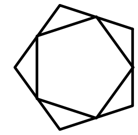
    (a) Undefined **(b)** $\frac{119}{169}$ **(c)** $-\frac{24}{13}$

    (d) $-\frac{119}{169}$ **(e)** None of the above

    SOLUTION **(b)**: Let $\alpha = \cos^{-1}\left(-\frac{12}{13}\right)$. Then $\cos(\alpha) = -\frac{12}{13}$, and

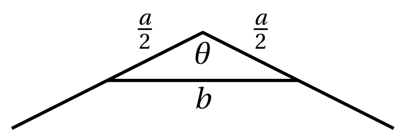
$$\cos(2\alpha) = 2\cos^2\alpha - 1 = 2\frac{144}{169} - 1 = \frac{119}{169}$$

14. Let θ be the angle measure of a regular polygon with n vertices, where n is a fixed natural number larger than 2. One such polygon is inscribed in another so that the vertices of the inner polygon are the midpoints of the edges of the outer one (the figure gives an example for $n = 5$). If the side of the outer polygon is a units and that of the inner polygon is b units, which of the following is true?



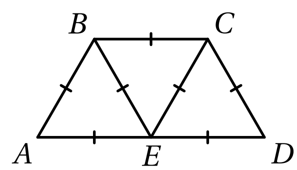
(a) $\cos\theta = \dfrac{a^2 - 2b^2}{a^2}$ **(b)** $\cos\theta = \dfrac{2a^2 - b^2}{2a^2}$ **(c)** $\sin(\theta/2) = \dfrac{a}{2b}$

(d) $\tan\theta = \dfrac{b}{\sqrt{a^2 - b^2}}$ **(e)** None of the above



SOLUTION **(a)**: Drawing one vertex of the outer polygon and the surrounding two vertices of the inner polygon we get an isosceles triangle with two sides of length $a/2$ surrounding the angle θ and a side of length b opposite to θ. By the law of cosines, we have $2\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)\cos\theta = \left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2 - b^2$. Hence $\cos\theta = \dfrac{a^2 - 2b^2}{a^2}$.

15. Given that the trapezoid $ABCD$ has area 12, find the length of the segment $\overline{AE}$. (Congruent line segments are marked with small perpendicular dashes through the middle of the segment).



(a) $\sqrt{6}$ **(b)** $\frac{4}{\sqrt{3}}$ **(c)** $\frac{4}{\sqrt[4]{3}}$ **(d)** 6 **(e)** None of the above

SOLUTION **(c)**: Let x be the length of the segment $\overline{AE}$. The height of the trapezoid is the height of the equilateral triangle ABE, which is $\frac{\sqrt{3}}{2}x$. The area of the trapezoid is then $\dfrac{3\sqrt{3}}{4}x^2$, which has to be equal to 12.

16. Suppose $f(1) = 2$ and $f(5) = 3$. If for every $k \geq 3$, $f(k) = f(k-2) - f(k-1)$, find $f(2)$.

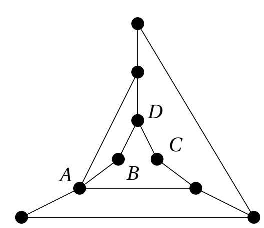
(a) $\frac{7}{5}$ **(b)** $\frac{5}{2}$ **(c)** -3 **(d)** $\frac{1}{3}$ **(e)** None of the above

SOLUTION **(d)**: Let $f(2) = y$. Then

$$\begin{aligned} f(3) &= f(1) - f(2) = 2 - y \\ f(4) &= f(2) - f(3) = y - (2 - y) = 2y - 2 \\ f(5) &= f(3) - f(4) = 2 - y - (2y - 2) = 4 - 3y \end{aligned}$$

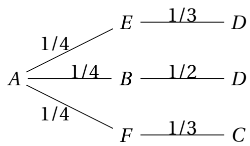
But $f(5) = 3$ so we have $4 - 3y = 3$, or $y = 1/3$.

17. A peg located at one vertex on the corresponding graph will be moved to another vertex that is connected to its current location by an edge; the selection of the new position is otherwise totally random. If the peg is currently at the vertex A, what is the probability that it will end up after two moves at an interior vertex (one of vertices B, C or D)?



(a) 7/24 **(b)** 1/2 **(c)** 3/10 **(d)** 5/24

(e) None of the above



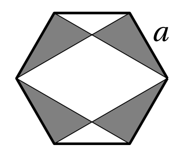
SOLUTION **(a)**: Denote the vertex directly above D as E, and the vertex directly to the right of A as F. The adjacent tree diagram shows all possible two moves that start from A and end at B, C or D, together with the probability of each. Hence the total probability is $(1/4)(1/3)+(1/4)(1/2)+(1/4)(1/3)=7/24$.

18. Suppose that $f(x)\ge 0$ for all x, and a, b are positive numbers. Given that the area of the region bounded by the graph of $y=f(x)\sin(x)$, $0\le x\le\pi$, and the x-axis is 2 units, and that the area bounded by the graph $y=af(bx)\sin(bx)$, $0\le x\le\pi/b$, and the x-axis is 8 units, which of the following holds?

(a) $a=4$ **(b)** $a=2$ **(c)** $\frac{a}{b}=4$ **(d)** $\frac{b}{a}=4$ **(e)** None of the above

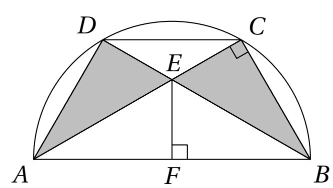
SOLUTION **(c)**: The area under $y=af(bx)\sin(bx)$ is obtained from that under $y=f(x)\sin(x)$ by stretching vertically by factor a and shrinking horizontally by factor b. Hence we have $8=(\dfrac{a}{b})(2)$.

19. In the figure is a regular hexagon of side length a. The ratio of the shaded area to that of the entire hexagon is equal to



(a) $\frac{1}{3}$ **(b)** $\frac{2}{3}$ **(c)** $\frac{4}{9}$ **(d)** $\frac{1}{4}$ **(e)** None of the above

SOLUTION **(c)**: The area of the hexagon can be computed by dividing it into six congruent triangles (by joining the center to the vertices). Each triangle will be equilateral of side length a. Hence the area of the hexagon is $6(\dfrac{1}{2})(a)(\dfrac{\sqrt{3}}{2}a)=\dfrac{3\sqrt{3}a^2}{2}$. The shaded region consists of four congruent triangles, and can be realized from the symmetry (or by simple angle geometry) that each is 60°-30° right triangle with the side a opposite to the 60° angle. Hence the shaded region has area $4(\dfrac{1}{2})(a)(\dfrac{a}{\sqrt{3}})=\dfrac{2a^2}{\sqrt{3}}$. Thus the ratio of the areas is $\frac{2a^2}{\sqrt{3}}\div\frac{3\sqrt{3}a^2}{2}=\frac{4}{9}$.



Alternative solution: The ratio of the shaded area to the area of the hexagon is the same as the ratio of the shaded area in the upper half of the hexagon to the area of the upper half (see the picture).

The triangles AEB and CED are similar, with the length of AB being twice the length of CD, and so the area of the triangle ABE is 4 times the area of CDE. Since the triangle BCD is isosceles, the angle CBD is congruent to CDB which is congruent to ABD. The point C lies on a circle with diameter AB, which makes the angle ACB a right angle. Therefore the triangles BCE and BEF are congruent. In a similar way you can show that the triangle ADE is congruent to AEF. Therefore the shaded area is the same as the area of triangle ABE.

So the area of the upper half of the hexagon is 9 times the area of triangle CDE, while the shaded area is only 4 times the area of DCE. Therefore the ratio is 4/9.

20. Let a be a positive real number and k a natural number. Let f be a polynomial of degree 11 with a positive leading coefficient whose roots are $a, a\pm1, a\pm2, a\pm3, a\pm4, a\pm5$, and let $g(T) = (f(T))^2$. If the leading coefficient of g is a while the coefficient of T^{21} in g is $-44a$, what is $f(0)$?

(a) 0 **(b)** 11 **(c)** 1 **(d)** 11! **(e)** None of the above

SOLUTION **(a)**: The leading coefficient of f is $\sqrt{a}$. Thus $f(T) = \sqrt{a}\prod_{i=-5}^{5}(T-(a+i))$. Thus the T^{21} coefficient of g is $-2a$ times the sum of the roots of f; i.e. $(-2a)(11a) = -22a^2$. Equating this to $-44a$, we get $a = 2$. In particular, 0 (which is $a-2$) is a root of f. Now $f(0)$ is $\sqrt{a}$ times the product of the roots of f. Hence $f(0) = 0$.

21. Let $a = 5^{39}$, $b = 2^{62}$, $c = 6^{26}$, and $d = 3^{52}$. Which of the following inequalities holds?

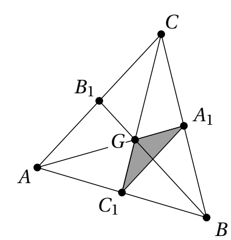
(a) $b < c < a < d$ **(b)** $c < a < d < b$ **(c)** $c < d < b < a$

(d) $b < c < d < a$ **(e)** None of the above

SOLUTION **(d)**: The answer is justifiable by the following sequence of inequalities.

$$\begin{aligned} b &= 2^{62} = 2^{26}\cdot 2^{36} = 2^{26}\cdot 8^{12} < 2^{26}\cdot 9^{12} = 2^{26}\cdot 3^{24} < 2^{26}\cdot 3^{26} = c \\ &< 3^{26}\cdot 3^{26} = 3^{52} = d = 9^{26} < 10^{26} = 2^{26}\cdot 5^{26} = 4^{13}\cdot 5^{26} < 5^{13+26} = a \end{aligned}$$

22. Assume that the area of triangle ABC is $4\,\text{ft}^2$. Let A_1, B_1 and C_1 be the midpoints of the sides BC, AC and AB, respectively, and let G be the point of intersection of AA_1, BB_1 and CC_1. Then the area of triangle A_1C_1G is



(a) $\frac{4}{9}$ **(b)** $\frac{1}{3}$ **(c)** $\frac{1}{12}$ **(d)** $\frac{1}{16}$ **(e)** None of the above

SOLUTION **(b)**: Since $\overline{CC_1}$ is a median in the triangle ABC, the triangles C_1BC and ABC have the same height, and the base of C_1BC is half of the base ABC. Therefore $A_{\triangle C_1BC} = \frac{1}{2}A_{\triangle ABC}$. Similarly, since $\overline{A_1C_1}$ is a median in the triangle C_1BC, $A_{\triangle C_1A_1C} = \frac{1}{2}A_{\triangle C_1BC}$. Since $CG : GC_1 = 2:1$, $A_{\triangle A_1GC_1} = \frac{1}{3}A_{\triangle C_1A_1C}$. Altogether

$$A_{\triangle A_1GC_1} = \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2}A_{\triangle ABC} = \frac{1}{12}(4) = \frac{1}{3}.$$

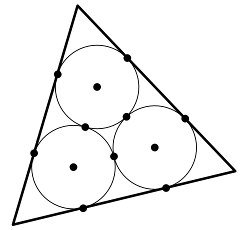
Alternative solution: Let $x = A_{\triangle A_1C_1G}$. Note that the segments $\overline{AC}$ and $\overline{A_1C_1}$ are parallel, which implies that the triangles CAG and C_1A_1G are similar. This together with the fact that $AC - 2A_1C_1$ implies that the area of the triangle ACG is $4x$.

The area of the trapezoid AC_1A_1C is 3 (it is 3/4 of the area of the triangle ABC), and the area of each of the triangles ACC_1 and ACA_1 is 2 (half of the area of the triangle ABC). The triangle ACG is exactly the overlap of the triangles ACC_1 and ACA_1, therefore the area of the trapezoid AC_1A_1C can be expressed as the sum of the areas of the triangles ACC_1, ACA_1 and A_1C_1G minus the area of the triangle ACG. This gives us an equation

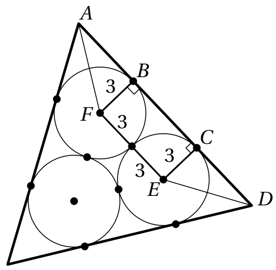
$$3 = 2 + 2 + x - 4x$$

which means that $x = 1/3$.

23. Three circles with radius 3 are inscribed in an equilateral triangle as shown in the picture. The side of the triangle is

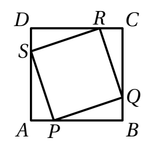


(a) 12 **(b)** $6\left(1+\sqrt{3}\right)$ **(c)** $3+3\sqrt{3}$ **(d)** 18 **(e)** None of the above



SOLUTION **(b)**: Because $ECBF$ is a rectangle, the length BC is 6. The triangles ABF and CDE are 30°-60°-90° triangles, therefore the lengths of AB and CD are both $3\sqrt{3}$. Together, the length AD is $3\sqrt{3}+6+3\sqrt{3}$.

24. Let $ABCD$ be a square with a side a and let the points P, Q, R, S be on the sides AB, BC, CD, DA respectively, and such that $AP = BQ = CR = DS = \frac{1}{3}a$. Let $PQ = b$. Then $\dfrac{a}{b}$ is



(a) $\dfrac{3}{5}$ **(b)** $\dfrac{1}{3\sqrt{5}}$ **(c)** $\dfrac{5}{\sqrt{3}}$ **(d)** $\dfrac{3}{\sqrt{5}}$ **(e)** None of the above

SOLUTION **(d)**: Denote the length of AP by x. Then the length of PB is $2x$, $a = 3x$ and, using Pythagorean theorem, $b = \sqrt{(2x)^2 + x^2} = x\sqrt{5}$. Thus

$$\frac{a}{b} = \frac{3x}{x\sqrt{5}} = \frac{3}{\sqrt{5}}.$$

25. The expression $4\sin^3\alpha - 3\sin\alpha + \sin 3\alpha$ equals

(a) -2 **(b)** 0 **(c)** 2 **(d)** $6\sin\alpha$ **(e)** None of the above

SOLUTION **(b)**: Using addition and double angle formulas together with the Pythagorean identity $\sin^2\alpha + \cos^2\alpha = 1$,

$$\begin{aligned}
\sin 3\alpha &= \sin(2\alpha+\alpha)\\
&= \sin\alpha\cos 2\alpha + \cos\alpha\sin 2\alpha\\
&= \sin\alpha\left(1-2\sin^2\alpha\right) + \cos\alpha 2\sin\alpha\cos\alpha\\
&= \sin\alpha - 2\sin^3\alpha + 2\sin\alpha\cos^2\alpha\\
&= \sin\alpha - 2\sin^3\alpha + 2\sin\alpha(1-\sin^2\alpha)\\
&= \sin\alpha - 2\sin^3\alpha + 2\sin\alpha - 2\sin^3\alpha\\
&= 3\sin\alpha - 4\sin^3\alpha
\end{aligned}$$

Therefore $4\sin^3\alpha - 3\sin\alpha + \sin 3\alpha = 4\sin^3\alpha - 3\sin\alpha + 3\sin\alpha - 4\sin^3\alpha = 0$.